Applied Mathematics for Engineers
Midterm 2

Code : *MAT 210* Acad.Year : *2018-19* Semester : *Fall* Date : *16.12.2018* Time : *10:40* Duration : *110 min*	Last Name : Name : Solution Student ID : Signature : 6 QUESTIONS ON 5 PAGES TOTAL 100 POINTS	# :

P1. (20)	P2. (20)	P3. (20)	P4. (20)	P5. (20)		Total. (100)

1. *(10×2=20pts)* Short answer questions.

(A) How do you check if a force on a truss system is balanced ($\mathbf{f} = \mathbf{f}_b$)?

Test if $\underline{f}$ is in column space of B:

→ Try to solve $B\underline{w} = \underline{f}$ (Divide)

(B) How do you check if a force on a truss system is purely unbalanced ($\mathbf{f} = \mathbf{f}_m$)?

Test if f is in nullspace of $A = B^T$:

→ Check $B^T \underline{f} \overset{??}{=} 0$ (Multiply)

(C) How do you check if a truss system is stable using the force balance matrix, B?

Test if any vectors are $\perp$ columns of B

⇔ Find nullspace of $A = B^T$.

Reduce $A \to U$ (with row swaps & scaling)

⇔ Stable if each column has a pivot.

(D) How do you check if a truss system has extra bars using the elongation matrix, A?

Test if each bar gives a pivot in matrix.

Reduce $A \to U$

Extra bar ⇔ Row without pivot

Reduce $A^T = B \to U$

Extra bar ⇔ col. without pivot

(E) How are the elongation matrix and stiffness matrix of spring systems related?

$$K = A^T C A$$

stiffness matrix — spring const. — elongation mat.

(F) What does the transpose of the elongation matrix A^T compute?

$$A^T \underline{w} = \underline{f}$$

It computes force on masses from tension/compression in springs.

(G) Give one property of the numbers in Markov matrices.

- All numbers are $0 \le x \le 1$
- Sum of numbers in each column is 1

(H) Give one property of the eigenvalues of Markov matrices.

- $\lambda = 1$ is always an eigenvalue
- All eigenvalues are $-1 \le \lambda \le 1$

(I) How do you compute the angular frequencies of the fundamental modes of an oscillating spring system?

$$\omega = \sqrt{\lambda} \quad \left\{ \begin{array}{l} \text{where } \lambda \text{ are the} \\ \text{eigenvalues of } M^{-1}K \end{array} \right.$$

(J) A spring system oscillates with fundamental mode $\lambda = 2$, $v = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$. If the first mass has amplitude 1, what is the amplitude of the second mass?

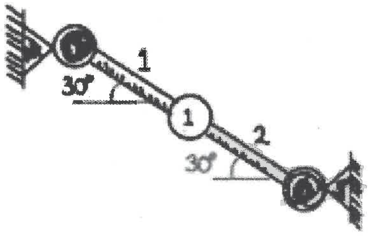
$$v = \begin{bmatrix} -2 \\ 3 \end{bmatrix} \xrightarrow{\text{rescale}} \begin{bmatrix} 1 \\ -3/2 \end{bmatrix}$$

Amplitude of mass #1

Amplitude of mass #2

Amplitude of mass #2 $= 3/2$ (oscillates in opposite dir)

2. *(4+2+3+3=12pts)* Consider the truss system to the right.



(A) Write the force balance matrix B.

$$B = \begin{array}{cc} & \begin{array}{cc}\text{Bar \#1} & \text{Bar \#2}\end{array} \\ \text{Node \#1}\begin{array}{c}x\\ y\end{array} & \begin{bmatrix}\cos 30 & -\cos 30\\ -\sin 30 & \sin 30\end{bmatrix}\end{array} = \begin{bmatrix}\sqrt{3}/2 & -\sqrt{3}/2\\ -1/2 & 1/2\end{bmatrix}$$

(B) Write the elongation matrix A.

$$A = B^T = \begin{bmatrix}\sqrt{3}/2 & -1/2\\ -\sqrt{3}/2 & 1/2\end{bmatrix}$$

(C) Write one force that is balanced $\mathbf{f}_\mathrm{b}$.

$\underline{\underline{f}}_b$ = any force of form $B\underline{\underline{w}}$

$$\text{e.g. } \begin{bmatrix}\sqrt{3}/2 & -\sqrt{3}/2\\ -1/2 & 1/2\end{bmatrix}\begin{bmatrix}2\\ 0\end{bmatrix} = \begin{bmatrix}\sqrt{3}\\ -1\end{bmatrix}$$

(D) Write one purely unbalanced force $\mathbf{f}_\mathrm{m}$.

$\underline{\underline{f}}_m$ = any force in null space A

$$\begin{bmatrix}\sqrt{3}/2 & -1/2\\ -\sqrt{3}/2 & 1/2\end{bmatrix} \to \begin{bmatrix}\sqrt{3} & -1\\ -\sqrt{3} & 1\end{bmatrix} \to \begin{array}{cc}\text{pivot} & \text{free}\\ \hline \sqrt{3} & -1\\ 0 & 0\end{array}$$

$$f_m = \begin{bmatrix}1\\ \sqrt{3}\end{bmatrix}$$

3. *(2+2+4=8pts)* Reducing a matrix (with row swaps and scaling) gives

$$A = \begin{bmatrix} 6 & 0 & 6 & -4 & 2\\ -9 & 0 & -9 & 6 & -3\\ 2 & 0 & 2 & 2 & 4\\ 3 & 0 & 3 & -2 & 1\end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 & 0 & 1\\ 0 & 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{bmatrix} = U$$

Give bases for the following.

(A) Row space of A.

Rowspace A = Rowspace U

Basis = $[1\ 0\ 1\ 0\ 1]$, $[0\ 0\ 0\ 1\ 1]$

(B) Column space of A.

Column space A = {columns of A with pivots in U}

$$\text{Basis} = \begin{bmatrix}6\\ -9\\ 2\\ 3\end{bmatrix}, \begin{bmatrix}-4\\ 6\\ 2\\ -2\end{bmatrix}$$

(C) Null space of A.

Null space A = Nullspace U

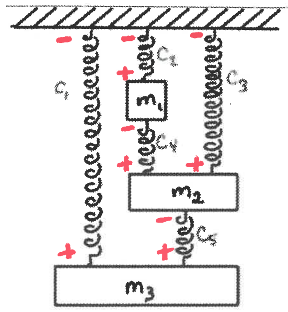
$$\begin{array}{ccccc} \text{pivot} & \text{free} & \text{free} & \text{pivot} & \text{free}\\ \hline 1 & 0 & 1 & 0 & 1\\ 0 & 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{array}$$

$$\begin{array}{ccccc} 0 & 1 & 0 & 0 & 0\\ \hline 1 & 0 & 1 & 0 & 1\\ 0 & 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{array} \begin{array}{l} \\ \to 0\\ \to 0\\ \\ \\ \end{array} \qquad \begin{array}{ccccc} -1 & 0 & 0 & -1 & 1\\ \hline 1 & 0 & 1 & 0 & 1\\ 0 & 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{array} \begin{array}{l} \\ \to 0\\ \to 0\\ \\ \\ \end{array}$$

$$\begin{array}{ccccc} -1 & 0 & 1 & 0 & 0\\ \hline 1 & 0 & 1 & 0 & 1\\ 0 & 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{array} \begin{array}{l} \\ \to 0\\ \to 0\\ \\ \\ \end{array}$$

$$\text{Basis} = \begin{bmatrix}0\\ 1\\ 0\\ 0\\ 0\end{bmatrix}, \begin{bmatrix}-1\\ 0\\ 1\\ 0\\ 0\end{bmatrix}, \begin{bmatrix}-1\\ 0\\ 0\\ -1\\ 1\end{bmatrix}$$

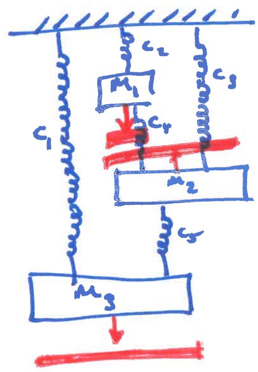
4. *(4+4+6+6=20pts)* Consider the spring system pictured to the right with spring constants $c_1 = 3$, $c_2 = 1$, $c_3 = 2$, $c_4 = 2$, $c_5 = 1$.
(Let down be the positive direction.)



(A) Write the elongation matrix A.

$$A = \begin{array}{c} \begin{array}{ccc} M_1 & M_2 & M_3 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{array}$$

(B) Compute the elongation caused by the displacement $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$.



$$\underline{e} = A\,\underline{u} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \\ -2 \\ 3 \end{bmatrix}$$

(C) Write the stiffness matrix K.

$$K = \begin{bmatrix} c_2 + c_4 & -c_4 & 0 \\ -c_4 & c_3 + c_4 + c_5 & -c_5 \\ 0 & -c_5 & c_1 + c_5 \end{bmatrix} = \begin{bmatrix} 3 & -2 & 0 \\ -2 & 5 & -1 \\ 0 & -1 & 4 \end{bmatrix}$$

(D) Compute the force required to cause the displacement $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$.

$$\underline{f} = K\,\underline{u} = \begin{bmatrix} 3 & -2 & 0 \\ -2 & 5 & -1 \\ 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ -9 \\ 9 \end{bmatrix}$$

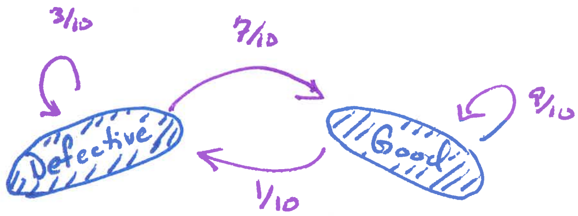
5. *(2+2+1+1+6+8=20pts)* The output of a machine is either defective or good. After making a defective product, the probability of the next product also being defective is $3/10$. After making a good product, the probability of the next product being defective is $1/10$.

(A) Write the transition matrix for the system using the state vector $\mathbf{v} = \begin{bmatrix} \text{Defective} \\ \text{Good} \end{bmatrix}$

$$\begin{array}{ccccc} & \text{Def} & \text{Good} & & \\ & \downarrow & \downarrow & & \\ K = & \frac{3}{10} & \frac{1}{10} & \to & \text{Def.} \\ & 1-\frac{3}{10} & 1-\frac{1}{10} & \to & \text{Good} \end{array}$$

$$= \begin{bmatrix} 3/10 & 1/10 \\ 7/10 & 9/10 \end{bmatrix}$$

(B) Draw the transition graph for the system.



(C) If the first product is good, then what is the probability of the **second** product being good?

$$\underline{v}_0 = \left\{\begin{matrix} \text{good} \\ \text{output} \end{matrix}\right\} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{array}{l} \leftarrow \text{Def.} \\ \leftarrow \text{Good} \end{array}$$

$$\underline{v}_1 = K\underline{v}_0 = \begin{bmatrix} 3/10 & 1/10 \\ 7/10 & 9/10 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/10 \\ 9/10 \end{bmatrix}$$

$$\boxed{\text{Prob.(good)} = 9/10 \qquad (90\%)}$$

(D) If the first product is good, then what is the probability of the **third** product being good?

$$\underline{v}_2 = K v_1$$

$$= \begin{bmatrix} 3/10 & 1/10 \\ 7/10 & 9/10 \end{bmatrix}\begin{bmatrix} 1/10 \\ 9/10 \end{bmatrix} = \begin{bmatrix} 12/100 \\ 88/100 \end{bmatrix}$$

$$\boxed{\text{Prob(good)} = 88/100 = 22/25 \qquad (88\%)}$$

The transition matrix has eigenvalues and eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 7 \end{bmatrix} \text{ with } \lambda_1 = 1 \qquad \text{and} \qquad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \text{ with } \lambda_2 = 2/10$$

(E) What is the over-all probability of good products from the machine?

(Long term prob. of good products)

$\lambda = 1$ eigenvector is $\begin{bmatrix} 1 \\ 7 \end{bmatrix} \xrightarrow{\text{Rescale to be prob.}} \frac{1}{1+7}\begin{bmatrix} 1 \\ 7 \end{bmatrix} = \begin{bmatrix} 1/8 \\ 7/8 \end{bmatrix}$

$$\boxed{\text{Prob(good)} = 7/8}$$

(F) After an initial good product, what is the probability of the nth later product to be good?

① Write initial state as sum of eigenvectors

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = a\begin{bmatrix} 1 \\ 7 \end{bmatrix} + b\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ 7 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 1 & 1 \\ -7 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/8 \\ 1/8 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/8 \\ 7/8 \end{bmatrix} + \begin{bmatrix} -1/8 \\ 1/8 \end{bmatrix}$$

② Multiply by eigenvalues

$$K^n\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/8 \\ 7/8 \end{bmatrix} + (2/10)^n\begin{bmatrix} -1/8 \\ 1/8 \end{bmatrix}$$

$$\boxed{\begin{aligned} \text{Prob(good)} &= 7/8 + (2/10)^n(1/8) \\ &= 1/8\left[7 + (2/10)^n\right] \end{aligned}}$$

6. *(5×4=20pts)* The following parts are about eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$

(In the parts below do not **compute the characteristic equation or any determinants!)**

(A) Find the eigenvalue for $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

Solve $A\underline{v} = \lambda \underline{v}$

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \lambda\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\boxed{\lambda = 3}$$

(B) Show that $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is not an eigenvector.

Show that $A\underline{v} = \lambda \underline{v}$ has no solution.

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 4 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} \lambda \\ \lambda \\ \lambda \end{bmatrix} = \lambda\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Impossible!

(C) Find the eigenvector for $\lambda = 1$.

Solve $\left(A - \begin{smallmatrix} 1 \text{ on} \\ \text{diag} \end{smallmatrix}\right)\underline{v} = \underline{0}$ by reducing $\left(A - \begin{smallmatrix} 1 \text{ on} \\ \text{diag} \end{smallmatrix}\right) \longrightarrow U$ (with row swaps & scaling)

$$\begin{bmatrix} 1-1 & 1 & -1 \\ 1 & 2-1 & 1 \\ 1 & 0 & 3-1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \xrightarrow{\text{swap}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ \cancel{1} & 0 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & \cancel{1} & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

pivot pivot

free

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\underline{v} = \underline{0} \qquad \underline{v} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

$$\boxed{\underline{v} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}}$$

(D) Show that $\lambda = -1$ is not an eigenvalue.

Show that $\left(A - \begin{smallmatrix} (-1) \text{ on} \\ \text{diag} \end{smallmatrix}\right)\underline{v} = \underline{0}$ has only $\underline{v} = \underline{0}$ solution.

$$\begin{bmatrix} 1-(-1) & 1 & -1 \\ 1 & 2-(-1) & 1 \\ 1 & 0 & 3-(-1) \end{bmatrix}$$

$$\|$$

$$\begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & 1 \\ 1 & 0 & 4 \end{bmatrix} \xrightarrow{\text{swap}} \begin{bmatrix} 1 & 0 & 4 \\ \cancel{1} & 3 & 1 \\ \cancel{2} & 1 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 4 \\ 0 & 3 & -3 \\ 0 & 1 & -9 \end{bmatrix} \xrightarrow{\text{scale}} \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -1 \\ 0 & \cancel{1} & -9 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -1 \\ 0 & 0 & -8 \end{bmatrix}$$

pivot pivot

$$\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -1 \\ 0 & 0 & -8 \end{bmatrix}\underline{v} = \underline{0} \quad \text{has only solution } \underline{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$